

Approximate bound state solutions of Dirac equation with Hulthén potential including Coulomb-like tensor potential

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(Dated: March 26, 2010)

Abstract

We solve the Dirac equation approximately for the attractive scalar $S(r)$ and repulsive vector $V(r)$ Hulthén potentials including a Coulomb-like tensor potential with arbitrary spin-orbit coupling quantum number κ . In the framework of the spin and pseudospin symmetric concept, we obtain the analytic energy spectrum and the corresponding two-component upper- and lower-spinors of the two Dirac particles by means of the Nikiforov-Uvarov method in closed form. The limit of zero tensor coupling and the non-relativistic solution are obtained. The energy spectrum for various levels is presented for several κ values under the condition of exact spin symmetry in the presence or absence of tensor coupling.

Keywords: Dirac equation, spin and pseudospin symmetry, bound states, Tensor potential, Hulthén potential, Nikiforov-Uvarov method.

PACS numbers: 03.65.Pm; 03.65.Ge; 02.30.Gp

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I. INTRODUCTION

Within the framework of the Dirac equation the spin symmetry arises if the magnitude of the attractive Lorentz scalar potential $S(r)$ and the time-component repulsive vector potential are nearly equal, $S(r) \sim V(r)$ in nuclei (*i.e.*, when the difference potential $\Delta(r) = V(r) - S(r) = C_s = \text{constant}$). However, the pseudospin symmetry occurs when $S(r) \sim -V(r)$ are nearly equal (*i.e.*, when the sum potential $\Sigma(r) = V(r) + S(r) = C_{ps} = \text{constant}$) [1-3]. The bound states of nucleons seem to be sensitive to some mixtures of these potentials. The cases $\Delta(r) = 0$ and $\Sigma(r) = 0$ correspond to $SU(2)$ symmetries of the Dirac Hamiltonian [3]. The spin symmetry is relevant for mesons [4]. The pseudospin symmetry concept has been applied to many systems in nuclear physics and related areas [2-7]. Further, it is used to explain features of deformed nuclei [8], the super-deformation [9] and to establish an effective nuclear shell-model scheme [5,6,10]. The pseudospin symmetry appeared in nuclear physics refers to a quasi-degeneracy of the single-nucleon doublets and can be characterized with the non-relativistic quantum numbers $(n, l, j = l + 1/2)$ and $(n - 1, l + 2, j = l + 3/2)$, where n , l and j are the single-nucleon radial, orbital and total angular momentum quantum numbers for a single particle, respectively [5,6]. The total angular momentum is given as $j = \tilde{l} + \tilde{s}$, where $\tilde{l} = l + 1$ is a pseudo-angular momentum and $\tilde{s} = 1/2$ is a pseudospin angular momentum. In real nuclei, the pseudospin symmetry is only an approximation and the quality of approximation depends on the pseudo-centrifugal potential and pseudospin orbital potential [11]. The Dirac Hamiltonian with vector and scalar potentials quadratic in space coordinates has been studied [12]. It was shown that the Dirac equation can be solved exactly for the cases $\Delta(r) = 0$ and $\Sigma(r) = 0$. In addition, the linear tensor potential and mixture of quadratic scalar and vector potentials are studied [13]. It is shown that a linear tensor potential with quadratic $\Delta(r)$ or $\Sigma(r)$ generates a harmonic-oscillator-like second order differential equation which can be solved analytically. Recently, Akcay [14,15] has shown that the Dirac equation for scalar and vector quadratic potentials including the Coulomb-like tensor potential with the spin and pseudospin symmetries can be solved exactly. These results in Dirac equation with quadratic potential plus a centrifugal-like potential can also be solved analytically. Tensor coupling potentials are added as spin-orbit coupling terms to the Dirac Hamiltonian by making the substitution $\vec{p} \rightarrow \vec{p} - im\omega\beta.\hat{r}U(r)$ [16,17]. Tensor couplings and exactly solvable tensor potential have been used to investigate

nuclear properties [18-20] and have also some physical applications [21,22].

In the past years, there has been much interest in the solution of the relativistic Dirac and Klein-Gordon equations [1,12,23-26]. For instance, some authors have solved these equations for several physical potentials, such as the Woods-Saxon potential [23-26], the Morse potential [27], the Hulthén potential [28], the Eckart potential [29-31], the Pöschl-Teller potential [32,33] and the Scarf-type potential [34], etc.

Recently, many works have been done to solve the Dirac equation so to obtain the energy equation and the two-component spinor wave functions. Jia *et al* [35] employed an improved approximation scheme to deal with the new centrifugal spin-orbit term $\kappa(\kappa + 1)r^{-2}$ in the second order differential equation that results from the Dirac equation and to solve it for the generalized Pöschl-Teller potential for arbitrary spin-orbit quantum number κ . Zhang *et al* [36] solved the Dirac equation with equal Scarf-type scalar and vector potentials by the method of the supersymmetric (SUSY) quantum mechanics, shape invariance approach and by alternative methods. Zou *et al* [37] solved the Dirac equation with equal Eckart scalar and vector potentials in terms of SUSY quantum mechanical method, shape invariance approach and function analysis method. Wei and Dong [38] obtained approximately the analytical bound state solutions of the Dirac equation with the Manning-Rosen for arbitrary κ . Thylwe [39] presented the approach inspired by amplitude-phase method in analyzing the radial Dirac equation to calculate phase shifts by including the spin- and pseudo-spin symmetries of relativistic spectra. Alhaidari [40] solved Dirac equation by separation of variables in spherical coordinates for a large class of non-central electromagnetic potentials. Berkdemir and Sever [41] investigated systematically the pseudospin symmetric solution of the Dirac equation for spin 1/2 particles moving within the Kratzer potential connected with an angle-dependent potential. Recently, we have also solved the spin and pseudospin symmetric Dirac equation with arbitrary spin-orbit centrifugal term for generalized Woods-Saxon potential [42] and Rosen-Morse potential [43] by means of the Nikiforov-Uvarov (NU) method.

In this paper, it is worth to investigate the solution of the Dirac equation for scalar and vector Hulthén potential for $\Delta(r)$ or $\Sigma(r)$ together with Coulomb shape tensor coupling potential which can be solved analytically by using an improved approximation scheme introduced in Refs. [44,45] to deal with the resulting centrifugal and pseudo-centrifugal terms $\kappa(\kappa \pm 1)r^{-2}$. The Coulomb-like tensor potential preserves the form of the Hulthén potential but generates a new spin-orbit centrifugal terms $\Lambda(\Lambda \pm 1)r^{-2}$, where Λ is a new

spin-orbit quantum number. This provides a possibility for generating a different form of spin-orbit coupling which might have some physical applications.

The Hulthén potential, widely used for the description of the nucleon-heavy nucleus interactions, takes the following form (see [44] and the references therein):

$$V_H(r) = -\frac{V_0}{e^{r/r_0} - 1}, \quad r_0 = \delta^{-1}, \quad V_0 = Ze^2\delta, \quad (1)$$

where V_0 is the potential depth, δ is the screening range parameter and r_0 represents the spatial range. If the potential is used for atoms, then $V_0 = Z\delta$ (in the relativistic units $\hbar = c = e = 1$), where Z is identified as the atomic number. The Hulthén potential behaves like the Coulomb potential near the origin (i.e., $r \rightarrow 0$ or $r \ll r_0$) $V_C(r) = -Ze^2/r$, but decreases exponentially in the asymptotic region when $r \gg 0$, so its capacity for bound states is smaller than the Coulomb potential. This potential has been applied to a number of areas such as nuclear and particle physics [46-48], atomic physics [49,50], molecular physics [51,52] and chemical physics [53], etc.

In the presence of the spin and pseudospin symmetry, we investigate the bound state energy eigenvalues and corresponding upper and lower spinor wave functions for arbitrary spin-orbit κ quantum number in the framework of the NU method [54-56]. We also show that the spin and pseudospin symmetric Dirac solutions can be reduced to the $S(r) = V(r)$ and $S(r) = -V(r)$ in the cases of exact spin symmetry limitation $\Delta(r) = 0$ and pseudospin symmetry limitation $\Sigma(r) = 0$, respectively. Furthermore, the solutions obtained for the Dirac equation can be easily reduced to the Schrödinger solutions when a parametric transformation is applied.

In what follows, we first review the NU method and present a parametric generalization in Sect. 2. In the presence of spin and pseudo-spin symmetries, we obtain the bound state solutions of the Dirac equation with scalar and vector Hulthén potentials including the Coulomb-like tensor interaction, the limit of zero tensor coupling and the non-relativistic limits by applying a suitable transformation in Sect. 3. The relevant concluding remarks are given in Sect. 4.

II. THE NIKIFOROV-UVAROV METHOD

The NU method [54] is briefly outlined here. It is based on solving the second-order differential equation of hypergeometric-type by means of special orthogonal functions:

$$\psi''(r) + \frac{\tilde{\tau}(r)}{\sigma(r)}\psi'(r) + \frac{\tilde{\sigma}(r)}{\sigma^2(r)}\psi(r) = 0, \quad (2)$$

where $\sigma(r)$ and $\tilde{\sigma}(r)$ are polynomials at most of second-degree, $\tilde{\tau}(r)$ is a first-degree polynomial and $\psi(r)$ is function of the hypergeometric type. In order to find a particular solution for Eq. (2), we choose $\psi(r)$ as follows:

$$\psi(r) = \phi(r)y(r), \quad (3)$$

which leads to a new second-order hypergeometric-type equation of the form

$$y''(r) + A(r)y'(r) + B(r)y(r) = 0, \quad (4)$$

with $A(r)$ and $B(r)$ are taken to be $\tau(r)/\sigma(r)$ and $\tilde{\sigma}(r)/\sigma^2(r)$, respectively, where $\tilde{\sigma}(r) = \lambda\sigma(r)$, λ is a constant and $\tau(r)$ is a polynomial of degree at most one. If we impose that $\phi(r)$ satisfies the following logarithmic equation

$$\frac{\phi'(r)}{\phi(r)} = \frac{\pi(r)}{\sigma(r)}, \quad (5)$$

then we obtain

$$\tau(r) = \tilde{\tau}(r) + 2\pi(r), \quad \tau'(r) < 0, \quad (6)$$

where $\pi(r)$ is a polynomial of order at most one. Making use of $\phi''(r)/\phi(r) = (\phi'(r)/\phi(r))' + (\phi'(r)/\phi(r))^2$, we can reduce Eq. (4) into another hypergeometric-type:

$$\sigma(r)y''(r) + \tau(r)y'(r) + \lambda y(r) = 0, \quad (7)$$

and the quadratic equation for the polynomial $\pi(r)$,

$$\pi^2(r) + [\tilde{\tau}(r) - \sigma'(r)]\pi(r) + \tilde{\sigma}(r) - k\sigma(r) = 0, \quad (8)$$

where

$$k = \lambda - \pi'(r). \quad (9)$$

The solution of the above quadratic equation for $\pi(r)$ yields

$$\pi(r) = \frac{1}{2} [\sigma'(r) - \tilde{\tau}(r)] \pm \sqrt{\frac{1}{4} [\sigma'(r) - \tilde{\tau}(r)]^2 - \tilde{\sigma}(r) + k\sigma(r)}, \quad (10)$$

and the weight function can be obtained via

$$[\sigma(r)\rho(r)]' = \tau(r)\rho(r), \quad (11)$$

where the prime denotes the differentiation with respect to r . The expression under the square root sign in Eq. (10) can be arranged as a polynomial of second order where its discriminant is zero. Hence, an equation for k is being obtained. After solving such an equation, the k values are determined through the NU method. One, however, is looking for a family of solutions corresponding to

$$\lambda = \lambda_n = -n\tau'(r) - \frac{1}{2}n(n-1)\sigma''(r), \quad n = 0, 1, 2, \dots \quad (12)$$

The $y(r) = y_n(r)$ which is a polynomial of degree n can be expressed in terms of the Rodrigues relation:

$$y_n(r) = \frac{B_n}{\rho(r)} \frac{d^n}{dr^n} [\sigma^n(r)\rho(r)], \quad (13)$$

where B_n is the normalization constant and the weight function $\rho(r)$ in (13) is the solution of the differential equation (11).

In addition, an eigenvalue solution through the NU method can be set up from the relationship between λ and λ_n by means of Eqs. (9) and (12).

For a more simple application of the method, we develop a parametric generalization of the NU method valid for any central and non-central exponential-type potential by making change of the independent variables $z = z(r)$. Thus, we obtain another generalized hypergeometric equation

$$[z(1 - c_3z)]^2 w''(z) + [z(1 - c_3z)(c_1 - c_2z)] w'(z) + (-q_2z^2 + q_1z - q_0) w(z) = 0, \quad (14)$$

when compared with Eq. (1) yields

$$\tilde{\tau}(z) = c_1 - c_2z, \quad \sigma(z) = z(1 - c_3z), \quad \tilde{\sigma}(z) = -q_2z^2 + q_1z - q_0, \quad (15)$$

where the parameters c_j and q_j ($j = 0, 1, 2$) are to be determined during the solution procedure. Thus, by following the method, we can also obtain all the analytic polynomials and their relevant constants necessary for the solution of a radial wave equation. Hence, these analytical expressions are displayed in Appendix A.

III. DIRAC EQUATION WITH A TENSOR COUPLING

In spherical coordinates, the Dirac equation for fermionic massive spin-1/2 particles moving in attractive scalar $S(r)$, repulsive vector $V(r)$ and a tensor $U(r)$ potentials reads as (in relativistic units $\hbar = c = 1$) [13,57]:

$$[\alpha \cdot \mathbf{p} + \beta (M + S(r)) + V(r) - i\beta\alpha \cdot \hat{\mathbf{r}}U(r) - E] \psi_{n\kappa}(\mathbf{r}) = 0, \quad \psi_{n\kappa}(\mathbf{r}) = \psi(r, \theta, \phi), \quad (16)$$

where E is the relativistic energy of the system, M is the mass of the fermionic particle, $\mathbf{p} = -i\nabla$ is the three-dimensional momentum operator, and α and β are 4×4 Dirac matrices, which have the following forms, respectively,

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix}, \quad (17)$$

where \mathbf{I} denotes the 2×2 identity matrix and σ are three-vector Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (18)$$

For a particle in a spherical (central) field, the total angular momentum operator \mathbf{J} and the spin-orbit matrix operator $\hat{\mathbf{K}} = -\beta(\sigma \cdot \mathbf{L} + \mathbf{I})$ commute with the Dirac Hamiltonian, where \mathbf{L} is the orbital angular momentum operator. For a given total angular momentum j , the eigenvalues of $\hat{\mathbf{K}}$ are $\kappa = -(j + 1/2)$ for aligned spin ($s_{1/2}$, $p_{3/2}$, *etc.*) and $\kappa = j + 1/2$ for unaligned spin ($p_{1/2}$, $d_{3/2}$, *etc.*). The spinor wavefunctions can be classified according to the radial quantum number n and the spin-orbit quantum number κ and can be written using the Pauli-Dirac representation:

$$\psi_{n\kappa}(r, \theta, \phi) = \frac{1}{r} \begin{pmatrix} F_{n\kappa}(r) Y_{jm}^l(\theta, \phi) \\ iG_{n\kappa}(r) Y_{jm}^{\tilde{l}}(\theta, \phi) \end{pmatrix}, \quad (19)$$

where $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ are the radial wave functions of the upper- and lower-spinor components, respectively, $Y_{jm}^l(\theta, \phi)$ and $Y_{jm}^{\tilde{l}}(\theta, \phi)$ are the spherical harmonic functions coupled to the total angular momentum j and its projection m on the z axis. The orbital and pseudo-orbital angular momentum quantum numbers for spin symmetry l and pseudospin symmetry \tilde{l} refer to the upper- and lower-components, respectively. For a given spin-orbit quantum number $\kappa = \pm 1, \pm 2, \dots$, the orbital angular momentum and pseudo-orbital angular momentum are given by $l = |\kappa + 1/2| - 1/2$ and $\tilde{l} = |\kappa - 1/2| - 1/2$, respectively.

The quasi-degenerate doublet structure can be expressed in terms of a pseudo-spin angular momentum $\tilde{s} = 1/2$ and pseudo-orbital angular momentum \tilde{l} which is defined as $\tilde{l} = l + 1$ for aligned spin $j = \tilde{l} - 1/2$ and $\tilde{l} = l - 1$ for unaligned spin $j = \tilde{l} + 1/2$. For example, $(3s_{1/2}, 2d_{3/2})$ and $(3\tilde{p}_{1/2}, 2\tilde{p}_{3/2})$ can be considered as pseudospin doublets.

Substituting Eq. (19) into Eq. (16) and using the following relations [57]

$$(\boldsymbol{\sigma} \cdot \mathbf{A})(\boldsymbol{\sigma} \cdot \mathbf{B}) = \mathbf{A} \cdot \mathbf{B} + i\boldsymbol{\sigma} \cdot (\mathbf{A} \times \mathbf{B}), \quad (20a)$$

$$(\boldsymbol{\sigma} \cdot \mathbf{P}) = \boldsymbol{\sigma} \cdot \hat{\mathbf{r}} \left(\hat{\mathbf{r}} \cdot \mathbf{P} + i \frac{\boldsymbol{\sigma} \cdot \mathbf{L}}{r} \right), \quad (20b)$$

and properties

$$(\boldsymbol{\sigma} \cdot \mathbf{L}) Y_{jm}^{\tilde{l}}(\theta, \phi) = (\kappa - 1) Y_{jm}^{\tilde{l}}(\theta, \phi), \quad (21a)$$

$$(\boldsymbol{\sigma} \cdot \mathbf{L}) Y_{jm}^l(\theta, \phi) = -(\kappa + 1) Y_{jm}^l(\theta, \phi), \quad (21b)$$

$$(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}) Y_{jm}^{\tilde{l}}(\theta, \phi) = -Y_{jm}^l(\theta, \phi), \quad (21c)$$

$$(\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}) Y_{jm}^l(\theta, \phi) = -Y_{jm}^{\tilde{l}}(\theta, \phi), \quad (21d)$$

we obtain the following two radial coupled Dirac equations for the spinor components:

$$\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r) \right) F_{n\kappa}(r) = (M^2 + E_{n\kappa} - \Delta(r)) G_{n\kappa}(r), \quad (22a)$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r) \right) G_{n\kappa}(r) = (M^2 - E_{n\kappa} + \Sigma(r)) F_{n\kappa}(r), \quad (22b)$$

where $\Delta(r) = V(r) - S(r)$ and $\Sigma(r) = V(r) + S(r)$ are the difference and sum potentials, respectively. By eliminating $G_{n\kappa}(r)$ in Eq. (22a) and $F_{n\kappa}(r)$ in Eq. (22b), we get two second-order non-linear differential equations for the upper and lower radial spinor components, respectively

$$\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa}{r}U(r) - \frac{dU(r)}{dr} - U^2(r) + \frac{\frac{d\Delta(r)}{dr}}{M + E_{n\kappa} - \Delta(r)} \left(\frac{d}{dr} + \frac{\kappa}{r} - U(r) \right) \right\} F_{n\kappa}(r) \\ = (M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r)) F_{n\kappa}(r), \quad (23)$$

$$\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} + \frac{2\kappa}{r}U(r) + \frac{dU(r)}{dr} - U^2(r) + \frac{\frac{d\Sigma(r)}{dr}}{M - E_{n\kappa} + \Sigma(r)} \left(\frac{d}{dr} - \frac{\kappa}{r} + U(r) \right) \right\} G_{n\kappa}(r) \\ = (M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r)) G_{n\kappa}(r), \quad (24)$$

where $\kappa(\kappa+1) = l(l+1)$ and $\kappa(\kappa-1) = \tilde{l}(\tilde{l}+1)$. The radial wave functions are required to satisfy the necessary boundary conditions, that is, $F_{n\kappa}(0) = G_{n\kappa}(0) = 0$ and $F_{n\kappa}(r) = G_{n\kappa}(r) \rightarrow 0$ at infinity.

At this stage, we take the $\Sigma(r)$ or $\Delta(r)$ the form of Hulthén potential (1) and the tensor potential in the form of the Coulomb-like interaction. Equations (23) and (24) can be solved exactly for $\kappa = 0, -1$ and $\kappa = 0, 1$, respectively, because of the spin-orbit centrifugal term.

A. Spin symmetric bound state solution

In this part we are taking the $\Sigma(r)$ as the Hulthén potential and $\Delta(r) = C_s = \text{constant}$ ($\frac{d\Delta(r)}{dr} = 0$), i.e.,

$$\Sigma(r) = V_H(r) = -\frac{\Sigma_0}{e^{r/r_0} - 1}, \quad \Sigma_0 = V_0, \quad (25)$$

In fact, when the limit of r_0 becomes infinity, then $\lim_{r_0 \rightarrow \infty} \Sigma(r) = \infty$, it tells us that the Dirac particle could not be trapped by the Hulthén potential, which does not have any bound state under the condition of spin symmetry. The Coulomb-like potential [58] for the tensor due to a charge $Z_a e$ interacting with a charge $Z_A e$, distributed uniformly over a sphere of radius R_c , is added,

$$U_{Coul}(r) = -\frac{H}{r}, \quad H = \frac{Z_a Z_A e^2}{4\pi\epsilon_0}, \quad r \geq R_c, \quad (26)$$

where $R_c = 7.78$ fm is the Coulomb radius, and Z_a and Z_A denote the charges of the projectile a and the target nuclei A , respectively. Under this symmetry, substituting Eqs. (25) and (26) into Eq. (23), the equation obtained for the upper radial spinor $F_{n\kappa}(r)$ becomes

$$\left[\frac{d^2}{dr^2} - \frac{\gamma}{r^2} + \frac{\beta}{(e^{r/r_0} - 1) r_0^2} + \frac{\mathcal{E}_{n\kappa}}{r_0^2} \right] F_{n\kappa}(r) = 0, \quad (27)$$

where

$$\gamma = \eta_k (\eta_k - 1), \quad \eta_k = \kappa + H + 1, \quad (28a)$$

$$\beta = r_0^2 (E_{n\kappa} + M - C_s) V_0 > 0, \quad (28b)$$

$$\mathcal{E}_{n\kappa} = r_0^2 (E_{n\kappa} - M) (E_{n\kappa} + M - C_s) \leq \gamma d_0, \quad M \geq E_{n\kappa}, \quad (28c)$$

with $\kappa = l$ and $\kappa = -(l+1)$ for $\kappa > 0$ and $\kappa < 0$, respectively. Also, the quantum condition in (27) is obtained from the finiteness of the solution at the origin point

$$F_{n\kappa}(0) = 0, \quad (29)$$

and at infinity

$$\lim_{r \rightarrow \infty} F_{n\kappa}(r) = 0, \quad (30)$$

The spin symmetric energy eigenvalues depend on n and κ , *i.e.*, $E_{n\kappa} = E(n, \kappa(\kappa + 1))$. In Eq. (25), the choice of $\Sigma(r) = 2V(r) \rightarrow V(r)$ as mentioned in Ref. [12] allows us to reduce the resulting relativistic solutions into their non-relativistic limits under appropriate transformations.

Equation (27) can not be solved exactly only for the case of $\kappa = 1$ due to the spin-orbit centrifugal term γr^{-2} which is expanded in terms of singular functions of e^{-r/r_0} compatible with the solvability of the problem for $r \ll r_0$. Now, since the orbital term r^{-2} is too singular, the validity of such approximation is limited only to very few of the lowest energy states. Therefore, to go to higher energy states one may attempt to solve the relativistic version of the problem in [44] like the Dirac equation (27). Therefore, we apply the approximation scheme derived in [44] for the centrifugal term which is valid for small screening parameter δ values (*i.e.*, for large r_0 values or $r \ll r_0$). It can be casted in the form [44]:

$$\frac{1}{r^2} \approx \frac{1}{r_0^2} \left[d_0 + \frac{1}{e^{r/r_0} - 1} + \frac{1}{(e^{r/r_0} - 1)^2} \right], \quad (31)$$

where the dimensionless constant $d_0 = 1/12$ is exact as reported in many recent works (*cf.* *e.g.*, [35,59-62]). Obviously, the above approximation to the centrifugal (pseudo-centrifugal) term turns to r^{-2} when the parameter r_0 goes to infinity (small screening parameter δ) as

$$\lim_{r_0 \rightarrow \infty} \left[\frac{1}{r_0^2} \left(d_0 + \frac{1}{e^{r/r_0} - 1} + \frac{1}{(e^{r/r_0} - 1)^2} \right) \right] = \frac{1}{r^2},$$

which shows that the usual approximation is the limit of our approximation (*cf.* *e.g.*, [44] and the references therein). In terms of the new variable $z(r) = e^{-r/r_0}$, which maps the interval $r \in (0, \infty)$ into $z \in (0, 1)$, and using the approximation in Eq. (31), then Eq. (27) transforms into rational functions as

$$\left\{ \frac{d^2}{dz^2} + \frac{(1-z)}{z(1-z)} \frac{d}{dz} - \frac{1}{z^2(1-z)^2} [q_2 z^2 - q_1 z + q_0] \right\} F_{n,\kappa}(z) = 0, \quad (32)$$

where

$$q_2 = \beta + \gamma d_0 - \mathcal{E}_{n\kappa}, \quad (33a)$$

$$q_1 = \beta + 2\gamma d_0 - \gamma - 2\mathcal{E}_{n\kappa}, \quad (33b)$$

$$q_0 = \gamma d_0 - \mathcal{E}_{n\kappa} = \lambda_{n\kappa}^2, \quad (33c)$$

and $\lambda_{n\kappa}$ must be a positive real parameter for the presence of bound states. We are looking for solutions in the form

$$F_{n,\kappa}(z) = z^\xi(1-z)^\varsigma f(z), \quad \xi > 0, \quad \varsigma \geq 1, \quad (34)$$

where ξ and ς are real positive parameters and the boundary conditions in Eqs. (29) and (30) are also satisfied. Then, by substituting (34) into (32) the following hypergeometric differential equation for $f(z)$ is obtained

$$z(1-z)\frac{d^2}{dz^2}f(z) + [2\xi + 1 - (2\xi + 2\varsigma + 1)z]\frac{d}{dz}f(z) - [(\xi + \varsigma)^2 - q_2]f(z) = 0, \quad (35)$$

Thus, the wave functions satisfying Eqs. (27), (30) and (34) are given by

$$F_{n,\kappa}(z) = \mathcal{N}_{n\kappa}(1-z)^{\eta_k} z^{\lambda_{n\kappa}} {}_2F_1(-n, n + 2\lambda_{n\kappa} + 2\eta_k + 1, 2\lambda_{n\kappa} + 1; z), \quad (36)$$

where

$$\eta_k = \frac{1}{2} + \sqrt{\frac{1}{4} + q_0 + q_2 - q_1} = \kappa + H + 1 \rightarrow \eta_k^\pm = \begin{cases} l + H + 1, & \kappa > 0, \\ -l + H, & \kappa < 0. \end{cases} \quad (37)$$

For the bound-state problem, the solutions (36) fulfill the boundary condition (29) when

$$\lambda_{n\kappa} + \eta_k - \sqrt{q_2} = -n, \quad n = 0, 1, 2, \dots, \quad (38)$$

where n is the number of the nodes of the radial wave functions. This also results in the following energy spectrum formula

$$\lambda_{n\kappa} = \frac{\beta - (n + \kappa + H + 1)^2}{2(n + \kappa + H + 1)}. \quad (39)$$

The last formula gives an equation for the energy levels. When we insert $\lambda_{n\kappa}$ and β into the above equation, we find the following transcendental energy spectrum formula

$$\begin{aligned} (E_{n\kappa} - M)(E_{n\kappa} + M - C_s) &= \frac{d_0(\kappa + H)(\kappa + H + 1)}{r_0^2} \\ &- \frac{1}{4} \left[\frac{r_0(E_{n\kappa} + M - C_s)V_0}{(n + \kappa + H + 1)} - \frac{(n + \kappa + H + 1)}{r_0} \right]^2, \\ n = 0, 1, 2, \dots, n_{\max} &= r_0 \sqrt{V_0(E_{n\kappa} + M - C_s)} - \kappa - H - 1, \end{aligned} \quad (40)$$

where n_{\max} being the largest integer which is less than $r_0 \sqrt{V_0(E_{n\kappa} + M - C_s)} - \kappa - H - 1$, $E_{n\kappa} > C_s - M$. In the limit of zero tensor couplings ($H = 0$), the spin aligned states ($\kappa < 0$)

demand that $n+1 \neq -\kappa$. However, such a condition is no longer exist for the spin unaligned states ($\kappa > 0$).

On the other hand, we may also apply the NU method, this can be done when Eq. (32) is compared with Eq. (14), the following polynomials are found

$$\tilde{\tau}(z) = 1 - z, \quad \sigma(z) = z(1 - z), \quad \tilde{\sigma}(z) = q_2 z^2 - q_1 z + q_0. \quad (41)$$

Furthermore, we can follow the parametric generalization model of the NU method given in Appendix A to obtain the specific values for the constants c_i ($i = 1, 2, \dots, 13$), the results are listed in Table 1 for the potential model under consideration. When the relations (A1-A4) of Appendix A together with the values of constants given in Table 1 are applied, the key polynomials take the following particular forms [63]:

$$\pi(z) = -\frac{z}{2} - \left[\left(\eta_k + \lambda_{n\kappa} - \frac{1}{2} \right) z - \lambda_{n\kappa} \right], \quad (42)$$

and

$$k = \beta - \gamma - 2\lambda_{n\kappa} \left(\eta_k - \frac{1}{2} \right), \quad (43)$$

for discrete bound state solutions. According to the NU method, we further obtain

$$\begin{aligned} \tau(z) &= 1 - 2 \left[\left(\eta_k + \lambda_{n\kappa} + \frac{1}{2} \right) z - \lambda_{n\kappa} \right], \\ \tau'(z) &= -2 \left(\eta_k + \lambda_{n\kappa} + \frac{1}{2} \right) < 0, \end{aligned} \quad (44)$$

with prime denotes the derivative with respect to z . Referring to Table 1 and applying the relation A5 of Appendix A, we obtain energy equation formula which is identical to Eq. (40).

In the limit of zero tensor coupling and under the usual approximation, $H = d_0 = C_s = 0$, $\kappa = l$ and $V_0 = Ze^2\delta$, Eq. (40) becomes

$$\sqrt{M^2 - E_{nl}^2} = \frac{Ze^2 (E_{nl} + M)}{2(n+l+1)} - \frac{(n+l+1)}{2}\delta, \quad |E_{nl}| < M,$$

which is completely identical to Eq. (68) of Ref. [64] for the solution of the Klein-Gordon equation with equally mixed scalar and vector Hulthén potentials, i.e., $V_0 = S_0$ ($\Sigma_0 = 0$). Furthermore, the last equation is also identical to Eq. (25) upon inserting $q = 1$, $V_0 = S_0$ and $\delta = \delta_{\pm} = l+1, -l$ in Eqs. (8) and (9) of Ref. [65]. Therefore, real solutions are possible only for $|E_{nl}| \leq M$ (i.e., bound states).

We now look at some special cases and relationships between our results and some other existing results in literature. For exact spin symmetry, $V_0 = S_0$ case or $C_s = 0$ and applying the following appropriate transformations $E_{nl} + M \simeq 2\mu$, $E_{nl} - M \simeq E_{nl}$ and $\kappa = l$, we obtain the following non-relativistic energy spectrum for the Hulthén potential including the Coulomb-like interaction,

$$E_{nl} = \frac{\delta^2}{2\mu} \left\{ (H + l)(H + l + 1) d_0 - \left[\frac{\mu Z e^2}{\delta(H + n + l + 1)} - \frac{(H + n + l + 1)}{2} \right]^2 \right\}, \quad (45)$$

leading to the following energy spectrum formula in the limit of zero tensor coupling [44]

$$E_{nl} = \frac{\delta^2}{2M} \left\{ l(l + 1) d_0 - \left[\frac{\mu Z e^2}{\delta(n + l + 1)} - \frac{(n + l + 1)}{2} \right]^2 \right\},$$

$$n = 0, 1, 2, \dots, n_{\max} = r_0 \sqrt{2\mu V_0} - l - 1, \quad (46)$$

which is identical to Eq. (34) in Ref. [44] for the solution of the Schrödinger equation with the Hulthén potential. The largest integer $n_{\max} \leq r_0 \sqrt{2\mu V_0} - l - 1$ which is identical to Eqs. (12) and (13) given in Ref. [66] after choosing $A = 2\mu r_0^2 V_0$, $\alpha = 1$ and $\alpha = 0$ for the Hulthén potential case, respectively. In fact, when the limit of δ becomes zero, then

$$\lim_{r_0 \rightarrow \infty} E_{nl} = -\frac{1}{2M} \frac{(\mu Z e^2)^2}{(n + l + 1)^2},$$

which is a spectrum resulting from a Coulombic field.

We apply now the relations (A6-A10) of Appendix A to calculate the corresponding wave functions as [63]

$$\rho(z) = z^{2\lambda_{n\kappa}} (1 - z)^{2\eta_{\kappa}}, \quad (47)$$

$$\phi(z) = z^{\lambda_{n\kappa}} (1 - z)^{\eta_{\kappa}}, \quad (48)$$

where $\eta_{\kappa} \geq 1$ and $\lambda_{n\kappa}$ must be a positive real parameter for the presence of bound states. Hence, we find

$$y_n(z) = P_n^{(2\lambda_{n\kappa}, 2\eta_{\kappa})}(1 - 2z), \quad z \in [0, 1], \quad (49)$$

with $P_n^{(\nu, \mu)}(x)$ is the Jacobi polynomial, where $\nu > -1$, $\mu > -1$ and defined for the argument $x \in [-1, 1]$. Using the relation $F_{n\kappa}(z) = \phi(z)y_n(z)$, we get the radial upper-spinor wave functions as

$$F_{n\kappa}(r) = \mathcal{N}_{n\kappa} \frac{n! \Gamma(2\lambda_{n\kappa} + 1)}{\Gamma(n + 2\lambda_{n\kappa} + 1)} e^{-(\lambda_{n\kappa}/r_0)r} (1 - e^{-r/r_0})^{\eta_{\kappa}} P_n^{(2\lambda_{n\kappa}, 2\eta_{\kappa})}(1 - 2e^{-r/r_0}). \quad (50)$$

Furthermore, we give the relation linking the hypergeometric function and the Jacobi polynomials (see formula 8.962.1) in [67]

$${}_2F_1 \left(-n, n + \nu + \mu + 1, \nu + 1; \frac{1-x}{2} \right) = \frac{n! \Gamma(\nu + 1)}{\Gamma(n + \nu + 1)} P_n^{(\nu, \mu)}(x), \quad (51)$$

to rewrite the radial wave functions as

$$F_{n\kappa}(r) = \mathcal{N}_{n\kappa} e^{-(\lambda_{n\kappa}/r_0)r} (1 - e^{-r/r_0})^{\eta_\kappa} {}_2F_1 \left(-n, n + 2\lambda_{n\kappa} + 2\eta_\kappa + 1, 2\lambda_{n\kappa} + 1; e^{-r/r_0} \right). \quad (52)$$

where the normalization constant is calculated in Appendix B in closed form.

Before presenting the corresponding lower-component $G_{n\kappa}(r)$, let us recall a recurrence relation of hypergeometric function,

$$\frac{d}{dz} \left[{}_2F_1(a; b; c; z) \right] = \left(\frac{ab}{c} \right) {}_2F_1(a+1; b+1; c+1; z), \quad (53)$$

where

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{p=0}^{\infty} \frac{\Gamma(a+p)\Gamma(b+p)}{\Gamma(c+p)} \frac{z^p}{p!}. \quad (54)$$

which is used to solve Eq. (22a) and present the corresponding lower component $G_{n\kappa}(r)$ as follows

$$\begin{aligned} G_{n\kappa}(r) &= \frac{1}{M + E_{n\kappa} - C_s} \left[\frac{d}{dr} + \frac{\kappa}{r} - U(r) \right] F_{n\kappa}(r) \\ &= \frac{\mathcal{N}_{n\kappa} e^{-(\lambda_{n\kappa}/r_0)r} (1 - e^{-r/r_0})^{\eta_\kappa}}{r_0 (M + E_{n\kappa} - C_s)} \left\{ \left[-\lambda_{n\kappa} + \eta_\kappa \frac{e^{-r/r_0}}{1 - e^{-r/r_0}} + \frac{(\kappa + H) r_0}{r} \right] \right. \\ &\quad \times {}_2F_1 \left(-n, n + 2\lambda_{n\kappa} + 2\eta_\kappa + 1, 2\lambda_{n\kappa} + 1; e^{-r/r_0} \right) \\ &\quad \left. - {}_2F_1 \left(-n + 1, n + 2(\lambda_{n\kappa} + \eta_\kappa + 1), 2(\lambda_{n\kappa} + 1); e^{-r/r_0} \right) \right\}. \end{aligned} \quad (55)$$

Here, we remark that the hypergeometric series ${}_2F_1 \left(-n, n + 2\lambda_{n\kappa} + 2\eta_\kappa + 1, 2\lambda_{n\kappa} + 1; e^{-r/r_0} \right)$ terminates for $n = 0$ and thus converges for all values of real parameters η_κ and $\lambda_{n\kappa}$. It is worthy to note that in the limit of r_0 goes to infinity then the Dirac spinor components become as

$$\lim_{r_0 \rightarrow \infty} F_{n\kappa}(r) = \lim_{r_0 \rightarrow \infty} G_{n\kappa}(r) = 0,$$

which show that these components become unbound.

For $C_s > M + E_{n\kappa}$ and $E_{n\kappa} < M$ or $C_s < M + E_{n\kappa}$ and $E_{n\kappa} > M$, we note that parameters given in Eq. (33c) turn to be imaginary, i.e., $\lambda_{n\kappa}^2 < 0$ in the s -state ($\kappa = -1$) case. As a

result, the condition of existing bound states are $\lambda_{n\kappa} > 0$ and $\eta_\kappa > 0$, that is to say, in the case of $C_s > M + E_{n\kappa}$ and $E_{n\kappa} < M$, bound-states do not exist for some quantum number κ such as the s -state ($\kappa = -1$). Of course, if these conditions are satisfied for existing bound-states, the energy equation and wave functions are the same as these given in Eqs. (45), (52) and (55).

In the spin-symmetric case, the numerical bound state energy spectrum is obtained from Eq. (40) with the choice of the following values for the parameters: $M = 10 \text{ fm}^{-1}$, $r_0 = 10 \text{ fm}$, $V_0 = 10 \text{ fm}^{-1}$ and $C_s = 10.1 \text{ fm}^{-1}$. Thus the approximated values for the energy states using the usual approximation scheme (31) (i.e., with $d_0 = 0$) are presented in Table 2 for states $n = 0, 1, 2$ and 3 and spin-orbit quantum number $\kappa = \pm 1, \pm 2, \pm 3$ and ± 4 values. Apparantly, the spin unaligned states ($\kappa > 0$), the orbital states ($l = |\kappa + 1/2| - 1/2$, $j = |\kappa| - 1/2$): $(1p_{1/2}, 0d_{3/2})$, $(2p_{1/2}, 1d_{3/2}, 0f_{5/2})$, $(3p_{1/2}, 2d_{3/2}, 1f_{5/2})$ and $(3d_{3/2}, 2f_{5/2})$ are found to be degenerate in the presence of tensor ($H \neq 0$) and in the limit of zero tensor couplings ($H = 0$). In addition, the energies of the $np_{1/2}$, $nd_{3/2}$ and $nf_{5/2}$ states for $H \neq 0$ are higher than the corresponding values for $H = 0$. Overmore, we reproduced the energy spectrum of Table 1 using the new approximation scheme (31) proposed in Ref. [44] through taking the dimensionless constant $d_0 \approx 1/12$ [35,59-62]. The energy levels presented in Table 2 are slightly different under our new approximation scheme. Further, the degeneracies in the above states have no longer exist. This is true for both Coulomb tensor ($H \neq 0$) and in the limit of zero tensor ($H = 0$) interactions.

B. Pseudospin symmetric bound state solutions

The exact pseudospin symmetry occurs in the Dirac equation when $S(r) \sim -V(r)$. In this part we will take the Hulthén potential for $\Delta(r)$, (i.e., $\frac{d\Sigma(r)}{dr} = 0$, or $\Sigma(r) = C_{ps} = \text{constant}$),

$$\Delta(r) = -\frac{\Delta_0}{e^{r/r_0} - 1}, \quad \Delta_0 = V_0, \quad (56)$$

where Δ_0 is a constant and the tensor potential as in Eq. (26). Thus, Eq. (24) can be reduced into the Schrödinger-type:

$$\left[\frac{d^2}{dr^2} - \frac{\tilde{\gamma}}{r^2} + \frac{\tilde{\beta}}{(e^{r/r_0} - 1) r_0^2} + \frac{\tilde{\mathcal{E}}_{nk}}{r_0^2} \right] F_{n\kappa}(r) = 0, \quad (57)$$

where

$$\tilde{\gamma} = \Lambda_k (\Lambda_k - 1), \quad \Lambda_k = \kappa + H, \quad (58a)$$

$$\tilde{\beta} = r_0^2 (E_{n\kappa} - M - C_{ps}) V_0, \quad (58b)$$

$$\tilde{\mathcal{E}}_{n\kappa} = r_0^2 (E_{n\kappa} + M) (E_{n\kappa} - M - C_{ps}) \leq \gamma d_0, \quad E_{n\kappa} \leq M + C_{ps}, \quad (58c)$$

where $\kappa = -\tilde{l}$ and $\kappa = \tilde{l} + 1$ for $\kappa < 0$ and $\kappa > 0$, respectively. In the pseudospin symmetry, the eigenstates with $\tilde{j} = \tilde{l} \pm \frac{1}{2}$ are degenerate for $\tilde{l} \neq 0$. The energy eigenvalues $E_{n\kappa}$ depend only on n and κ , *i.e.*, $E_{n\kappa} = E(n, \kappa(\kappa - 1))$. We now follow the previous procedures to obtain a differential equation for $G_{n,\kappa}(z)$,

$$\left\{ \frac{d^2}{dz^2} + \frac{(1-z)}{z(1-z)} \frac{d}{dz} - \frac{1}{z^2(1-z)^2} [p_2 z^2 - p_1 z + p_0] \right\} G_{n,\kappa}(z) = 0, \quad (59)$$

where

$$p_2 = \tilde{\beta} + \tilde{\gamma} d_0 - \tilde{\mathcal{E}}_{n\kappa}, \quad (60a)$$

$$p_1 = \tilde{\beta} + 2\tilde{\gamma} d_0 - \tilde{\gamma} - 2\tilde{\mathcal{E}}_{n\kappa}, \quad (60b)$$

$$p_0 = \tilde{\gamma} d_0 - \tilde{\mathcal{E}}_{n\kappa} = \tilde{\lambda}_{n\kappa}^2, \quad (60c)$$

and $\tilde{\lambda}_{n\kappa}$ must be a positive real parameter for real solution. Thus, the energy spectrum is then given by

$$\lambda_{n\kappa} = \frac{\tilde{\beta} - (n + \kappa + H)^2}{2(n + \kappa + H)}. \quad (61)$$

The last formula gives an equation for the energy spectrum. When we insert $\tilde{\lambda}_{n\kappa}$ and $\tilde{\beta}$ into the above equation, then we find the following transcendental energy equation

$$\begin{aligned} (E_{n\kappa} + M) (E_{n\kappa} - M - C_{ps}) &= \frac{d_0 (\kappa + H) (\kappa + H - 1)}{r_0^2} \\ &- \frac{1}{4} \left[\frac{r_0 (E_{n\kappa} - M - C_{ps}) V_0}{(n + \kappa + H)} - \frac{(n + \kappa + H)}{r_0} \right]^2, \\ n = 0, 1, 2, \dots, n_{\max} &= r_0 \sqrt{V_0 (E_{n\kappa} - M - C_s)} - \kappa - H, \end{aligned} \quad (62)$$

where n_{\max} being the largest integer which is less than $r_0 \sqrt{V_0 (E_{n\kappa} - M - C_s)} - \kappa - H$, where $E_{n\kappa} > M + C_s$. Obviously, the above equation can be reduced to Klein-Gordon solution when $C_{ps} = H = d_0 = 0$ and after inserting $q = 1$, $V_0 = S_0$ and $\delta = \delta_{\pm} = \tilde{l} + 1, -\tilde{l}$ in Eqs. (8), (9) and (25) of Ref. [65]. Furthermore, the wave functions can be written as

$$G_{n,\kappa}(z) = \mathcal{N}_{n\kappa} (1-z)^{\Lambda_k} z^{\tilde{\lambda}_{n\kappa}} {}_2F_1 \left(-n, n + 2\tilde{\lambda}_{n\kappa} + 2\Lambda_k + 1, 2\tilde{\lambda}_{n\kappa} + 1; z \right),$$

$$= \mathcal{N}_{n\kappa} \frac{n! \Gamma(2\tilde{\lambda}_{n\kappa} + 1)}{\Gamma(n + 2\tilde{\lambda}_{n\kappa} + 1)} e^{-(\tilde{\lambda}_{n\kappa}/r_0)r} (1 - e^{-r/r_0})^{\Lambda_\kappa} P_n^{(2\tilde{\lambda}_{n\kappa}, 2\Lambda_\kappa)} (1 - 2e^{-r/r_0}). \quad (63)$$

where

$$\Lambda_\kappa = \kappa + H, \quad \Lambda_k^\pm = \begin{cases} \tilde{l} + H + 1, & \kappa > 0, \\ -\tilde{l} + H, & \kappa < 0. \end{cases} \quad (64)$$

To avoid repetition, the solution of Eq. (24) can be found easily by applying appropriate parametric transformations. A first inspection for the relationship between the present set of parameters $(\tilde{\mathcal{E}}_{nk}, \tilde{\beta}, \Lambda_k)$ and the previous set $(\mathcal{E}_{nk}, \beta, \eta_\kappa)$ tells us that the energy solution, for pseudospin symmetry, can be obtained directly from those of the previous energy solution, in spin symmetric case, by applying the following appropriate parameter map [42,43]:

$$F_{n,\kappa}(r) \leftrightarrow G_{n,\kappa}(r), \quad E_{n,\kappa} \rightarrow -E_{n,\kappa}, \quad C_s \rightarrow -C_{ps}, \quad V(r) \rightarrow -V(r) \text{ (or } V_0 \rightarrow -V_0), \quad (65)$$

or simply we apply the following transformations:

$$F_{n,\kappa}(r) \leftrightarrow G_{n,\kappa}(r), \quad M \rightarrow -M, \quad C_s \rightarrow C_{ps}, \quad \eta_\kappa \rightarrow \Lambda_\kappa \text{ (or } \kappa + H + 1 \rightarrow \kappa + H, \quad \kappa + H \rightarrow \kappa + H - 1), \quad (66)$$

on Eqs. (40) and (50) to obtain Eqs. (62) and (63) for energy spectrum formula and wave functions, respectively.

IV. CONCLUDING REMARKS

In this work, we have obtained analytically the approximate energy equation and the corresponding wavefunctions of the Dirac equation for the Hulthén potential coupled with a Coulombic-like tensor under the conditions of the spin and pseudospin symmetry using the parametric generalization of the NU method. The introduced Coulombic-like tensor interaction generates additional centrifugal-like term $L(L+1)r^{-2}$. Therefore, in order to solve the resulting Schrödinger-like equation analytically, a new approximation scheme was used to approximate the new spin-coupling centrifugal term γr^{-2} arising from the Coulomb-like tensor which yields a closed form solution of the problem under consideration. The resulting solutions of the wavefunctions are written in terms of the orthogonal Jacobi polynomials or hypergeometric functions. Obviously, for exact spin when $S(r) = V(r)$ (*i.e.*, $C_s = 0$), the relativistic solution can be easily reduced to its non-relativistic limit by the choice of appropriate mapping transformations. Also, in the limit of zero tensor couplings, the

present results reduce to the well-known solutions of the Dirac equation for the usual Hulthén potential with arbitrary spin-orbit coupling quantum number κ .

Acknowledgments

The authors highly appreciate the very constructive comments and suggestions provided from the kind referee and editor. The support provided by the Scientific and Technological Research Council of Turkey (TÜBİTAK) is highly appreciated.

Appendix A: Parametric Generalization of the NU Method

We complement the theoretical parameterized formulation of the NU method for any arbitrary exponential potential by giving the essential polynomials, energy equation and wavefunctions together with their relevant constants as follows [42,43,63].

(i) The key NU polynomials:

$$\pi(z) = c_4 + c_5 z - [(\sqrt{c_9} + c_3 \sqrt{c_8}) z - \sqrt{c_8}], \quad (\text{A1})$$

$$k = -(c_7 + 2c_3 c_8) - 2\sqrt{c_8 c_9}. \quad (\text{A2})$$

$$\tau(z) = 1 - (c_2 - 2c_5) z - 2[(\sqrt{c_9} + c_3 \sqrt{c_8}) z - \sqrt{c_8}], \quad (\text{A3})$$

$$\tau'(z) = -2c_3 - 2(\sqrt{c_9} + c_3 \sqrt{c_8}) < 0, \quad (\text{A4})$$

(ii) The general energy equation:

$$(c_2 - c_3)n + c_3 n^2 - (2n + 1)c_5 + (2n + 1)(\sqrt{c_9} + c_3 \sqrt{c_8}) + c_7 + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0. \quad (\text{A5})$$

(iii) The general wavefunctions:

$$\rho(z) = z^{c_{10}}(1 - c_3 z)^{c_{11}}, \quad (\text{A6})$$

$$\phi(z) = z^{c_{12}}(1 - c_3 z)^{c_{13}}, \quad c_{12} > 0, \quad c_{13} > 0, \quad (\text{A7})$$

$$y_n(z) = P_n^{(c_{10}, c_{11})}(1 - 2c_3 z), \quad \mathbb{R}c_{10} > -1, \quad \mathbb{R}c_{11} > -1, \quad (\text{A8})$$

$$u(z) = \mathcal{N}_n z^{c_{12}}(1 - c_3 z)^{c_{13}} P_n^{(c_{10}, c_{11})}(1 - 2c_3 z), \quad (\text{A9})$$

where $P_n^{(\nu, \mu)}(x)$, $\mathbb{R}\nu > -1$, $\mathbb{R}\mu > -1$ and $x \in [-1, 1]$ are the Jacobi polynomials and \mathcal{N}_n is a normalization constants. Also, the above wave functions can be expressed in terms of the hypergeometric function as

$$u_{n\kappa}(z) = \mathcal{N}_{n\kappa} z^{c_{12}}(1 - c_3 z)^{c_{13}} {}_2F_1(-n, 1 + c_{10} + c_{11} + n; c_{10} + 1; c_3 z), \quad (\text{A10})$$

where $c_{12} > 0$, $c_{13} > 0$ and $z \in [0, 1/c_3]$.

(iv) The relevant constants:

$$\begin{aligned} c_4 &= \frac{1}{2}(1 - c_1), \quad c_5 = \frac{1}{2}(c_2 - 2c_3), \quad c_6 = c_5^2 + B_1, \\ c_7 &= 2c_4 c_5 - B_2, \quad c_8 = c_4^2 + B_3, \quad c_9 = c_3(c_7 + c_3 c_8) + c_6, \\ c_{10} &= c_1 + 2c_4 + 2\sqrt{c_8} - 1 > -1, \quad c_{11} = 1 - c_1 - 2c_4 + \frac{2}{c_3}\sqrt{c_9} > -1, \\ c_{12} &= c_4 + \sqrt{c_8} > 0, \quad c_{13} = -c_4 + \frac{1}{c_3}(\sqrt{c_9} - c_5) > 0. \end{aligned} \quad (\text{A11})$$

Appendix B: Normalization of the radial wavefunctions

The normalization constant $\mathcal{N}_{n\kappa}$ can be easily calculated in closed form. To do this, we start by using the normalization condition $\int_0^\infty [F_{n\kappa}(r)]^2 dr = 1$, and under the coordinate change $x = 1 - 2e^{-r/r_0}$, the normalization constant $\mathcal{N}_{n\kappa}$ in (50) is given by

$$\mathcal{N}_{n\kappa}^{-2} = \frac{r_0}{2} \left[\frac{n! \Gamma(2\lambda_{n\kappa} + 1)}{\Gamma(n + 2\lambda_{n\kappa} + 1)} \right]^2 \int_{-1}^1 \left(\frac{1-x}{2} \right)^{2\lambda_{n\kappa}-1} \left(\frac{1+x}{2} \right)^{2\eta_\kappa} [P_n^{(2\lambda_{n\kappa}, 2\eta_\kappa)}(x)]^2 dx, \quad (\text{B1})$$

where $\lambda_{n\kappa} > 0$ and $\eta_\kappa \geq 1$. The calculation of this integral can be done by writing one of the $2\eta_\kappa$ factors $(1+x)/2$ in the form

$$\frac{1+x}{2} = 1 - \frac{1-x}{2}, \quad (\text{B2})$$

and by making use of the following integral (see formula (7.391.5) in [67]):

$$\int_{-1}^1 (1-x)^{\nu-1} (1+x)^\mu [P_n^{(\nu, \mu)}(x)]^2 dx = \frac{2^{\nu+\mu} \Gamma(n+\nu+1) \Gamma(n+\mu+1)}{n! \nu \Gamma(n+\nu+\mu+1)}, \quad (\text{B3})$$

which is valid for $\Re \nu > 0$ and $\Re \mu > -1$. This leads to

$$\mathcal{N}_{n\kappa} = \frac{1}{\Gamma(2\lambda_{n\kappa} + 1)} \left[\frac{2}{r_0} \frac{\lambda_{n\kappa} \Gamma(n + 2\lambda_{n\kappa} + 1) \Gamma(n + 2\lambda_{n\kappa} + 2\eta_\kappa + 1)}{n! \Gamma(n + 2\eta_\kappa + 1)} \right]^{1/2}. \quad (\text{B4})$$

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TABLE I: The NU parametric constants useful in calculating the energy eigenvalues and eigenfunctions for the spin-symmetric Dirac equation.

Constant		Analytic value	
Constant		Analytic value	
c_1	1	c_2	1
c_3	1	c_4	0
c_5	$-1/2$	c_6	$q_2 + 1/4$
c_7	$-q_1$	c_8	$\lambda_{n\kappa}^2$
c_9	$(\eta_\kappa - 1/2)^2$	c_{10}	$2\lambda_{n\kappa}$
c_{11}	$2\eta_\kappa - 1$	c_{12}	$\lambda_{n\kappa}$
c_{13}	η_κ		

TABLE II: The usual approximation to the relativistic energy spectrum (in fm^{-1}) of the spin-symmetric Hulthén potential including the Coulomb coupling tensor of strength $H = 0$ and $H = 0.5$ for various values of n and κ .

n	l	$\kappa < 0$	(l, j)	$E_{n,\kappa<0}(H=0)^a$	$E_{n,\kappa<0}(H=0.5)^a$	$\kappa > 0$	(l, j)	$E_{n,\kappa>0}(H=0)^a$	$E_{n,\kappa>0}(H=0.5)^a$
0	0	-1	$0s_{1/2}$	—	0.101446652	—	—	—	—
1	0	-1	$1s_{1/2}$	0.1057848200	0.100388800	—	—	—	—
2	0	-1	$2s_{1/2}$	0.1231107030	0.136077055	—	—	—	—
3	0	-1	$3s_{1/2}$	0.1518922615	0.170536938	—	—	—	—
0	1	-2	$0p_{3/2}$	0.1057848200	0.101446652	1	$0p_{1/2}$	0.123110703	0.1360770550
1	1	-2	$1p_{3/2}$	—	0.101446652	1	$1p_{1/2}$	0.1518922615	0.1705369375
2	1	-2	$2p_{3/2}$	0.105784820	0.100388800	1	$2p_{1/2}$	0.191988313	0.2162202946
3	1	-2	$3p_{3/2}$	0.123110703	0.136077055	1	$3p_{1/2}$	0.243203547	0.2729055754
0	2	-3	$0d_{5/2}$	0.123110703	0.100388800	2	$0d_{3/2}$	0.1518922615	0.1705369375
1	2	-3	$1d_{5/2}$	0.105784820	0.101446652	2	$1d_{3/2}$	0.191988313	0.2162202946
2	2	-3	$2d_{5/2}$	—	0.101446652	2	$2d_{3/2}$	0.243203547	0.2729055754
3	2	-3	$3d_{5/2}$	0.105784820	0.100388800	2	$3d_{3/2}$	0.3052908168	0.3403207458
0	3	-4	$0f_{7/2}$	0.1518922615	0.136077055	3	$0f_{5/2}$	0.191988313	0.2162202946
1	3	-4	$1f_{7/2}$	0.123110703	0.100388800	3	$1f_{5/2}$	0.243203547	0.2729055754
2	3	-4	$2f_{7/2}$	0.105784820	0.101446652	3	$2f_{5/2}$	0.3052908168	0.3403207458
3	3	-4	$3f_{7/2}$	—	0.101446652	3	$3f_{5/2}$	0.3779539814	0.4181464025

^aWe have taken $d_0 = 0$ during these calculations.

TABLE III: The new approximation to the relativistic energy spectrum (in fm^{-1}) of the spin-symmetric Hulthén potential including the Coulomb coupling tensor of strength $H = 0$ and $H = 0.5$ for various values of n and κ .

n	l	$\kappa < 0$	(l, j)	$E_{n,\kappa < 0}(H = 0)^a$	$E_{n,\kappa < 0}(H = 0.5)^a$	$\kappa > 0$	(l, j)	$E_{n,\kappa > 0}(H = 0)^a$	$E_{n,\kappa > 0}(H = 0.5)^a$
0	0	-1	$0s_{1/2}$	—	0.1014318359	—	—	—	—
1	0	-1	$1s_{1/2}$	0.105784820	0.1004034771	—	—	—	—
2	0	-1	$2s_{1/2}$	0.123110703	0.1360623873	—	—	—	—
3	0	-1	$3s_{1/2}$	0.1518922615	0.1705222701	—	—	—	—
0	1	-2	$0p_{3/2}$	0.100057870	0.1000005174	1	$0p_{1/2}$	0.1232273855	0.1362956215
1	1	-2	$1p_{3/2}$	—	0.1014893336	1	$1p_{1/2}$	0.1014382946	0.1707561995
2	1	-2	$2p_{3/2}$	0.100057870	0.1130529719	1	$2p_{1/2}$	0.1026480928	0.2164398762
3	1	-2	$3p_{3/2}$	0.123227386	0.1010361207	1	$3p_{1/2}$	0.2433208010	0.2731253525
0	2	-3	$0d_{5/2}$	0.123457217	0.1132253472	2	$0d_{3/2}$	0.1522417493	0.17104639476
1	2	-3	$1d_{5/2}$	0.1061170006	0.10163993177	2	$1d_{3/2}$	0.1923389117	0.2167313327
2	2	-3	$2d_{5/2}$	—	0.10163993177	2	$2d_{3/2}$	0.2435547169	0.2734174979
3	2	-3	$3d_{5/2}$	0.1061170006	0.1132253472	2	$3d_{3/2}$	0.3056423517	0.3408332607
0	3	-4	$0f_{7/2}$	0.1525865150	0.13658291603	3	$0f_{5/2}$	0.1926867981	0.2171368804
1	3	-4	$1f_{7/2}$	0.1000082780	0.1135029417	3	$1f_{5/2}$	0.2439041281	0.2738248097
2	3	-4	$2f_{7/2}$	0.0995403787	0.09964377877	3	$2f_{5/2}$	0.3059926539	0.34124166206
3	3	-4	$3f_{7/2}$	—	0.09964377877	3	$3f_{5/2}$	0.3786566934	0.4190685307

^aWe have taken $d_0 = 0.0823058167837972$ during these calculations.